Prove of Important Limits

Q.1. $\lim_{h\to 0}\frac{1-\cos h}{h} = 0$

$=\lim_{x\to 0}\frac{1-\cos x}{x}.\frac{1+\cos x}{1+\cos x} = \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$

$= \lim_{x\to 0}\frac{\sin x}{x}.\frac{\sin x}{(1+\cos x)} = \lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\frac{\sin x}{(1+\cos x)}$

= 1 x 0/2 = 1x 0 = 0

Q.2. $\lim_{x\to 1}\frac{x^n-1}{x-1}$

Let x = 1 + h

$= \lim_{x\to 1}\frac{x^n-1}{x-1} = \lim_{h\to 0}\frac{(1+h)^n-1}{1+h-1} = \lim_{h\to 0}\frac{1+nh+C_2^n h^2+C_3^n h^3+\cdots\ldots-1}{h}$

$= \lim_{h\to 0}\frac{h(n+C_2^n h^1+C_3^n h^2+\cdots\ldots+C_{n-1}^n)}{h}$ = n

Q.3. $\lim_{x\to 0}\frac{a^x-1}{x}$

Sol[2]: let $a^x - I = y$ so that $y \to 0$ as $x \to 0$

and $a^x = I + y$

or $x\ln a = \ln(I + y)$, where ln stands for log to the base e

Now $\lim_{x\to 0}\frac{ax-1}{x} =$

$\lim_{y\to 0}\frac{y}{\frac{1}{\ln a}\ln(1+y)} = \lim_{y\to 0}\frac{\ln a}{\frac{1}{y}\ln(1+y)} = \lim_{y\to 0}\frac{\ln a}{\frac{1}{\ln}(1+y)^{\frac{1}{y}}}$

$\lim_{x\to 0} \frac{ax-1}{x} = lna \lim_{y\to 0} \frac{1}{\ln(1+y)^{1/y}} = lna \frac{1}{ln \lim_{y\to 0}(1+y)^{1/y}}$

$= lna \frac{1}{lne}$ = lna

$\lim_{x\to 0} \frac{ax-1}{x} = lna$

Q.4. $\lim_{n\to\infty}(1+ \frac{1}{n})^n = e$
x→∞

Sol.

$(1+\frac{1}{n})^n$ expend it by using binomial thermo.

$(1+\frac{1}{n})^n = 1 + n \frac{1}{n} + n \frac{(n-1)}{2!} . \frac{1}{n}2 + n \frac{(n-1)(n-2)}{3!} . \frac{1}{n}3 + n \frac{(n-1)(n-2)(x-3)}{4!} . \frac{1}{n}4 + \ldots\ldots\ldots\ldots\ldots$
$+ n \frac{(n-1)(n-2)(x-3)\_\_\_(n-r-1)}{25!} . \frac{1}{n}4$

$(1+\frac{1}{n})^n = 1+1+ \frac{1}{2!} (1- \frac{1}{n}) + \frac{1}{3!} (1- \frac{1}{n}) (1- \frac{2}{n}) + \frac{1}{4!} (1- \frac{1}{n}) (1- \frac{2}{n})$

Apply limits x→∞ on both sides

$\lim_{n\to\infty}(1+ \frac{1}{n})^n = \lim_{n\to\infty}[1 + 1 + \frac{1}{2!} (1 - \frac{1}{n}) + \frac{1}{3!} (1 - \frac{1}{n}) (1 - \frac{2}{n})]$
x→∞ x→∞

$= 1+1+ \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots\ldots\ldots$
x→∞

This is almost equal to corrected upto 6 decimals

$$\lim_{n\to\infty}(1+ \frac{1}{n})^n = 2.718282$$

$$\lim_{n\to\infty}(1+ \frac{1}{n})^n = e$$

Q.5. $\lim_{n\to\infty}(1+ n)^{1/n}$ = e

Proof:

We Know that $\lim_{n\to\infty}(1+\frac{1}{n})^{n}$=e

let t = 1/n $t \to 0 \ as \ n \to \infty$

therefore

$$\lim_{n\to\infty}(1+\frac{1}{n})^{n} = \lim_{t\to 0}(1+t)^{1/t} = e$$